Copolymers at Selective Interfaces: New Bounds on the Phase Diagram

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Abstract We investigate the phase diagram of disordered copolymers at the interface between two selective solvents, and in particular its weak-coupling behavior, encoded in the slope m_c of the critical line at the origin. We focus on the directed walk case, which has turned out to be, in spite of the apparent simplicity, extremely challenging. In mathematical terms, the partition function of such a model does not depend on all the details of the Markov chain that models the polymer, but only on the time elapsed between successive returns to zero and on whether the walk is in the upper or lower half plane between such returns. This observation leads to a natural generalization of the model, in terms of arbitrary laws of return times: the most interesting case being the one of return times with power law tails (with exponent $1 + \alpha$, $\alpha = 1/2$ in the case of the symmetric random walk). The main results we present here are:

(1) the improvement of the known result $1/(1 + \alpha) \le m_c \le 1$, as soon as $\alpha > 1$ for what concerns the upper bound, and down to $\alpha \approx 0.65$ for the lower bound.
(2) a proof of the fact that the critical curve lies strictly below the critical curve of the annealed model for every non-zero value of the coupling parameter.

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We also provide an argument that rigorously shows the strong dependence of the phase diagram on the details of the return probability (and not only on the tail behavior). Lower bounds are obtained by exhibiting a new localization strategy, while upper bounds are based on estimates of non-integer moments of the partition function.

Keywords Directed polymers · Disorder · Copolymers at selective interfaces · Rare-stretch strategies · Fractional moment estimates

1 Introduction

Copolymers (or heteropolymers) are chains of non-identical monomer units. We focus here on the case in which some of the monomer units have an affinity for a solvent A, while the affinity of the others is for a solvent B. Affinities—below we will call them *charges*—are fixed along the polymer chain and we will model them as quenched disorder. The medium in which the (co)polymer fluctuates is the one schematized in Fig. 1: the two solvents occupy half of the space and they are separated by a sharp (and flat) interface. Copolymer models have an extended literature, notably models based on self-avoiding walks have been studied (see e.g. [29] and references therein), but a very simple model, that turned out to be nevertheless extremely challenging, has been proposed in [14]. It is a two-dimensional, in fact a $(1 + 1)$-dimensional model in which the self-avoidance property is enforced by considering directed walks and of course the walk steps are the monomer units. The nth monomer of the chain carries a *random charge* ϵ_n (which corresponds to $(\omega_n + h)$ in formula (2.1) below), which can be either positive or negative. Here *random* refers to the fact that the charges are not placed in a homogeneous or periodic way along the chain, but they are (a realization of) a collection of independent and identically distributed (IID) random variables. The charge ϵ_n quantifies the chemical affinity of the nth monomer with the solvents so that the monomer has an energetic preference for being placed in the solvent A (for example, oil) if $\epsilon_n > 0$ and in solvent B (water) if $\epsilon_n < 0$. It is then intuitive that an energy-entropy competition takes place—maximizing the energy by placing as many monomers as possible into their preferred solvent, versus wandering away from the interface and gaining in entropy—and this leads to a (non-trivial, as we will see) localization-delocalization transition, when the temperature or the mean of ϵ_n is varied.

The localization/delocalization critical curve $h_c(\lambda)$, in the (λ, h) plane (cf. Proposition 2.5: λ is the coupling parameter and h is an asymmetry parameter controlling the mean of the disorder), has attracted much attention, both in the theoretical physics [9, 14, 22, 25, 26, 28] and in the mathematics literature [1–3, 5, 15, 17, 24]. A point that has to be stressed is that one can find in the physical literature some predictions on the phase diagram and notably one can find expressions, claimed to be exact, of the critical curve (see below for more precision on this). However, these expressions do not coincide and this fact strongly suggests that the understanding of this transition is very limited (this is confirmed by the lack of agreement on critical exponents too [9, 20, 22, 28]). In addition, the arguments in many of the theoretical physics papers that we have mentioned appear not to depend on the details of the return probabilities, but just on the tail, in fact the arguments are developed for some large-scale *equivalent* systems in which the local details are forgotten. One of the results we present below shows that the critical curve does depend on these details, and at times even in a very radical way. What (most probably) does not depend on the details of the polymer and of the disorder is the slope m_c of the critical curve at the origin, which corresponds to weak polymer-solvent coupling.

Therefore m_c is definitely a quantity of great interest for its *universal* character.

From a mathematical standpoint bounds on the critical line have been established in [3, 5], but they are not sufficiently precise to settle the controversies between the different physical predictions (a result on the critical behavior has been proven in [19]; it shows that the transition is in *great generality* at least of second order). These bounds in some very exceptional cases do coincide (but these are really marginal situations, not considered by physicists). In particular the upper bound follows from an annealing procedure [5]. Very recently one of us [27] has proven that at large coupling parameter and for unbounded disorder the bound in [5] can be improved. Such a result is based on estimates on the fractional moments of the partition function. Here we will go beyond this result and get upper bounds that hold for arbitrary coupling parameter and general charge distributions, still by estimating fractional moments and by adapting an idea first developed in the context of disordered pinning models [11]. Let us mention that a result close to ours has been recently obtained, using different techniques, by E. Bolthausen and F. den Hollander [6]. It should be pointed out that, for copolymer models, constrained annealing techniques have been applied at several instances (e.g. [21] and references therein), but it has been shown in [7] that they are useless to go beyond the bound in [5] on the critical line.

Our purpose is to improve also the lower bounds. Finding lower bounds on $h_c(\lambda)$ amounts to finding lower bounds on the partition function, and to this aim it is natural to try to guess what the most favorable polymer configurations are and to keep only those in the partition sum. We will refer to such a trajectory selection as to a (selection) *strategy*. However natural this idea may look, it is difficult to guess strategies which give non-trivial bounds. In [22], a real-space renormalization procedure was implemented by using a *rare-stretch strategy* which takes advantage of atypical regions in the disorder. For some time we have believed that the strategy of [22] would yield a correct description of the critical curve and the correct value of m_c, even if in mathematical terms such a strategy led only to a lower bound [3]. Such a belief was later shaken by accurate numerical simulations [8]. In this paper we present a new rare-stretch strategy which, although not optimal, improves in some situations the critical curve lower bounds [3] based on the renormalization approach of [22].

Remark 1.1 In the physical literature only models based on symmetric walks have been considered. However we find that considering more general models helps in a substantial way in devising new arguments of proof and in understanding the limitations of previous approaches. Besides, the *generalized copolymer model* (already introduced and studied in [15, 18]) is a natural, and very easily defined, disordered model where one can investigate the effect of disorder on systems for which the associated annealed model has a first order phase transition.

Finally, we mention that also the situation where the geometry of the regions occupied by the two solvents is more involved than just two half-planes has been considered, see for instance [10].

2 The Model and the Results

2.1 The Standard Copolymer Model

The model of copolymers close to a selective interface introduced in [14] is based on the simple random walk $S := \{S_n\}_{n=0,1,\dots}$, that is the Markov chain characterized by $S_0 = 0$

and the fact that the increment sequence $\{S_{n+1} - S_n\}_{n=0,1,\dots}$ is IID with $\mathbf{P}(S_1 = +1) = \mathbf{P}(S_1 = -1) = 1/2$. The partition function of the model is defined as

$$\mathcal{Z}_{N,\omega} := \mathbf{E}\left[\exp\left(\lambda \sum_{n=1}^{N} (\omega_n + h)\,\mathrm{sign}(S_n)\right)\right], \tag{2.1}$$

where N is a positive integer, $\mathbf{E}$ is the expectation with respect to the random walk trajectory, λ and h are two constants that can be chosen non-negative without loss of generality, and $\omega := \{\omega_n\}_{n=1,2,\dots}$ is a sequence of real numbers. Since $\mathrm{sign}(0)$ is a priori not defined we stipulate that $\mathrm{sign}(S_n) = \mathrm{sign}(S_{n-1})$ if $S_n = 0$ (this arbitrary choice appears natural once the process is *decomposed into excursions*, see below). Let us also remark that the symbol $\mathbf{E}$ just denotes the normalized sum over the 2^N trajectories of the simple random walk. Figure 1 may be of help in order to get some insight.

Definition 2.1 Unless otherwise stated, the sequence ω, referred to as *sequence of charges*, is chosen as a typical realization of an IID sequence of law $\mathbb{P}$. We assume $M(t) := \mathbb{E}[\exp(t\omega_1)] < \infty$ for every $t \in \mathbb{R}$, and that $\mathbb{E}[\omega_1] = 0$ and $\mathbb{E}[\omega_1^2] = 1$.

A useful observation about this model is that $\mathcal{Z}_{N,\omega}$ can be expressed in terms of the return times $\tau := \{\tau_j\}_{j=0,1,\dots}$ defined iteratively by $\tau_0 = 0$ and $\tau_{j+1} := \inf\{n > \tau_j : S_n = 0\}$.
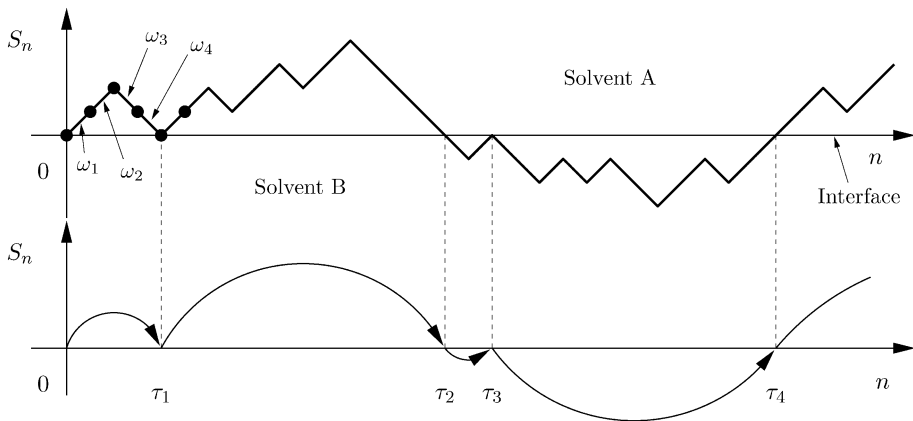

Fig. 1 In the top drawing one finds a trajectory of the standard copolymer model. The simple random walk bonds, that is the segments linking $(n-1, S_{n-1})$ and (n, S_n), are the monomers and they carry a charge ω_n that is drawn at random, but it is fixed once for all (quenched disorder), while the polymer fluctuates. Positively charged monomers are energetically rewarded if they lie in the *upper half-plane*, occupied by solvent A, while they are penalized if they lie in the *lower half-plane* (solvent B). The situation is reversed for negatively charged monomers. If the parameter h is not zero, the model is not symmetric under the exchange of the solvents. Note that the energetically favored trajectories are the ones that place most of the monomers in their favored solvent, but such trajectories are necessarily sticking close to the interface between the two solvents and they are therefore few compared to the trajectories that wander more freely. This leads to an energy-entropy competition between localized and delocalized trajectories. Observe also that the energy does not depend on the details of the trajectory between successive visits to 0, so that the model can be schematized like in the lower figure, that is simply in terms of $\tau_1 = 2$, $\tau_2 - \tau_1 = 6$, $\tau_3 - \tau_2 = 1$, $\tau_4 - \tau_3 = 6$, ... (that are distances between successive returns to zero of S_{2n}) and of the sign sequence $s_1 = +1$, $s_2 = +1$, $s_3 = -1$, $s_4 = -1$, $s_5 = +1$, ... The schematized version of the model can be easily generalized to arbitrary return laws $K(\cdot)$

Note that τ is a random walk itself: it has (positive and) IID increments $\{\tau_{j+1} - \tau_j\}_{j=0,1,\ldots}$ with a law $K(n) = \mathbf{P}(\tau_1 = n)$ that is explicitly written in combinatorial terms (and, notably, $\lim_{n\to\infty} n^{3/2} K(2n) = \sqrt{1/(4\pi)}$). By using a standard (probabilistic) terminology, we say that τ is a renewal sequence. Since $\text{sign}(S_n)$ is constant inside an *excursion* $\{\tau_j + 1, \tau_j + 2, \ldots, \tau_{j+1}\}$, it is natural to consider the sequence defined by $s_j = \text{sign}(S_{\tau_j})$, $j \geq 1$. An immediate consequence of the (strong) Markov property is that $s = \{s_j\}_{j=1,2,\ldots}$ is an IID sequence of symmetric random variables (taking of course only the values ± 1).

Before generalizing the model, let us immediately say that it is technically advantageous to work with a slightly different definition of the energy (and of the partition function (2.1)) of the model given by

$$Z_{N,\omega} := \mathbf{E}\left[\exp\left(-2\lambda \sum_{n=1}^{N} (\omega_n + h)\Delta_n \right) \right], \tag{2.2}$$

where $\Delta_n := [1 - \text{sign}(S_n)]/2$, so Δ_n is the indicator function that the nth-monomer is below the interface. Since the difference between the terms in the exponent of the expressions in (2.1) and (2.2) is just $\lambda \sum_{n=1}^{N} (\omega_n + h)$ (in particular, independent of S) the two models are actually the same and the asymptotic behaviors of $\mathscr{Z}_{N,\omega}$ and $Z_{N,\omega}$ are trivially related.

2.2 The Generalized Copolymer Model

The observation we have just made naturally leads to a generalization of the framework. We are in fact going to assume that $K(\cdot)$ is a discrete probability density on $\mathbb{N}$, so that $\sum_{n\in\mathbb{N}} K(n) = 1$, such that for some $\alpha > 0$

$$K(n) \overset{n\to\infty}{\sim} \frac{C_K}{n^{1+\alpha}}, \tag{2.3}$$

where $a_n \overset{n\to\infty}{\sim} b_n$ means that $\lim_{n\to\infty} a_n/b_n = 1$, and C_K is a positive constant. We define then a renewal process $\tau := \{\tau_0, \tau_1, \ldots\}$, i.e. a random walk with positive (and independent) increments such that $\tau_0 = 0$ and the sequence $\{\tau_j - \tau_{j-1}\}_{j=1,2,\ldots}$ is an IID sequence with law $\mathbf{P}(\tau_1 = n) = K(n)$ for every n. We also let $s = \{s_j\}_j$ be an IID sequence of symmetric random variables taking values ± 1, and for $n \in \mathbb{N}$ we define $\Delta_n = (1 - s_j)/2$ if $\tau_{j-1} < n \leq \tau_j$. The partition function of the generalized model is then again (2.2), where now $\mathbf{E}$ denotes the expectation with respect to τ and s.

Note that, strictly speaking, (2.3) is not compatible with the random walk choice which has only even return times. However, one can always redefine the return times of the simple random walk as $\tau/2$, and this is of course a trivial change (and $\alpha = 1/2$).

Remark 2.2 There is no difficulty in relaxing the assumption (2.3), for example by allowing *logarithmic* corrections to the asymptotic behavior. For the sake of simplicity, we are going to stick to assumption (2.3), with the notable exception of Remark 2.7 and Sect. 2.5.

2.3 The Free Energy and the Phase Diagram

The free energy (per unit length) of such a model is

$$\text{F}(\lambda, h) := \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log Z_{N,\omega}. \tag{2.4}$$

Actually, the $\mathbb{P}(d\omega)$-a.s. limit of the sequence of random variables $\{N^{-1}\log Z_{N,\omega}\}_N$ exists and coincides with $\textsc{f}(\lambda, h)$ (see, e.g., [15, Chap. 4]). We observe that $\textsc{f}(\lambda, h) \geq 0$ because

$$Z_{N,\omega} \geq \mathbf{E}\left[\exp\left(-2\lambda \sum_{n=1}^{N}(\omega_n + h)\Delta_n\right); \Delta_n = 0 \text{ for } n = 1, 2, \ldots, N\right]$$

$$= \mathbf{P}\left(\Delta_n = 0, n = 1, 2, \ldots, N\right) = \mathbf{P}(s_1 = +1, \tau_1 \geq N)$$

$$= \frac{1}{2}\sum_{n \geq N} K(n) \overset{N \to \infty}{\sim} \frac{C_K}{2\alpha N^\alpha}. \tag{2.5}$$

In more intuitive terms we say that the free energy in the delocalized regime is zero and we split the phase diagram according to

$$\mathcal{L} = \{(\lambda, h) : \textsc{f}(\lambda, h) > 0\} \quad \text{and} \quad \mathcal{D} = \{(\lambda, h) : \textsc{f}(\lambda, h) = 0\}. \tag{2.6}$$

Remark 2.3 Not surprisingly, the proof of the existence of the limit in (2.4) relies on super-additivity. Super-additivity turns out to be a very crucial tool for our arguments too, so let us stress that $\{\mathbb{E}\log Z_{N,\omega}\}_N$ is *not* a super-additive sequence. Rather one has to consider

$$Z_{N,\omega}^{\text{c}} := \mathbf{E}\left[\exp\left(-2\lambda \sum_{n=1}^{N}(\omega_n + h)\Delta_n\right); N \in \tau\right], \tag{2.7}$$

(c for *constrained*) where $N \in \tau$ simply means that $\tau_n = N$ for some n. It is easy to see that $\{\mathbb{E}\log Z_{N,\omega}^{\text{c}}\}_N$ is super-additive (and it is just a bit harder to see that $(1/N)\mathbb{E}\log Z_{N,\omega}^{\text{c}} - (1/N)\mathbb{E}\log Z_{N,\omega} \to 0$ as $N \to \infty$, so that, when talking of the free energy, we can safely switch between the two partition functions). Super-additivity says also that $\textsc{f}(\lambda, h)$ coincides with $\sup_N N^{-1}\mathbb{E}\log Z_{N,\omega}^{\text{c}}$ and this yields the following characterization:

$$\textsc{f}(\lambda, h) > 0 \quad \text{if and only if} \quad \text{there exists } N \text{ such that } \mathbb{E}\log Z_{N,\omega}^{\text{c}} > 0. \tag{2.8}$$

This is a powerful tool because it is a finite-volume criterion for localization. It has played a central role in a number of results, like [8, 17], and it will be, again, very important here.

Remark 2.4 In this work we leave aside any consideration on the copolymer path behavior and concentrate on free-energy properties. The fact that $(\lambda, h) \in \mathcal{L}$ (respectively $(\lambda, h) \in \mathcal{D}$) does correspond to localized (respectively, delocalized) behavior of the paths of the process has been addressed in depth elsewhere (e.g., [2, 17, 18] and [15, Chaps. 7 and 8]).

Convexity and monotonicity properties of the free energy entail a number of properties of the phase diagram that we sum up in the next statement (see [15, Chap. 6] for a proof, built on results proven in [3, 5]).

Proposition 2.5 (Existence of the critical curve) *There exists a continuous, strictly increasing function $\lambda \mapsto h_c(\lambda)$, satisfying $h_c(0) = 0$, such that $\mathcal{D} = \{(\lambda, h) : h \geq h_c(\lambda)\}$ (and of course $\mathcal{L} = \{(\lambda, h) : h < h_c(\lambda)\}$).*

This note addresses precisely the behavior of $h_c(\cdot)$. Let us first recall the known results: in the next statement we give bounds proven for the standard copolymer model in [5] for what

concerns the upper bound and in [3] for the lower bound (the straightforward adaptation of the arguments to cover the generalized model is detailed in [15, Chap. 6]). We set for $m > 0$

$$h^{(m)}(\lambda) := \frac{1}{2m\lambda} \log M(-2m\lambda),$$ (2.9)

where $M(t)$ is given in Definition 2.1 (note that $dh^{(m)}(\lambda)/d\lambda|_{\lambda=0} = m$).

Proposition 2.6 *For every choice of $K(\cdot)$ satisfying (2.3) we have*

$$h^{(1/(1+\alpha))}(\lambda) \le h_c(\lambda) \le h^{(1)}(\lambda),$$ (2.10)

for every $\lambda \ge 0$. As a consequence,

$$\frac{1}{1+\alpha} \le \liminf_{\lambda \searrow 0} \frac{h_c(\lambda)}{\lambda} \le \limsup_{\lambda \searrow 0} \frac{h_c(\lambda)}{\lambda} \le 1.$$ (2.11)

Remark 2.7 Upper and lower bounds in (2.10) coincide only if $\alpha = 0$, but assumption (2.3) requires $\alpha > 0$ in order to ensure that $K(\cdot)$ is normalizable. We can set $\alpha = 0$ if we accept to relax somewhat (2.3) (cf. Remark 2.2) and if we choose for example $K(n) = c/(n(\log n)^2)$ (c such that $\sum_n K(n) = 1$). Since (2.10) holds also when there are logarithmic corrections to the power law behavior of $K(\cdot)$, and in particular for $\alpha = 0$ (cf. [15, Chap. 6]), it is straightforward to see that $h_c(\lambda) = h^{(1)}(\lambda)$ for every $\lambda \ge 0$.

2.4 Weak-Coupling Limit, Rare-Stretch Strategy and a Look at the Literature

Much work has been done on the copolymer model, both in the physical and mathematical literature. In spite of this, the understanding of the model is still very limited. An important point has been set forth in [5], where it has been shown, for the case of the standard model (in particular, $\alpha = 1/2$) and for ω_1 binary random variable, that

$$\lim_{\gamma \searrow 0} \frac{1}{\gamma^2} F(\gamma\lambda, \gamma h) = \phi(\lambda, h),$$ (2.12)

where

$$\phi(\lambda, h) := \lim_{t \to \infty} \frac{1}{t} \mathbb{E} \log \mathbf{E} \left[\exp\left(-2\lambda \int_0^t \mathbf{1}_{B(t)<0} \, (d\beta(t) + h dt) \right) \right],$$ (2.13)

and B and β are two standard Brownian motions, respectively of law $\mathbf{P}$ and $\mathbb{P}$. Note that (2.13) is the partition function of a continuous copolymer model; like for the discrete case it is easy to see that $\phi(\lambda, h) \ge 0$ and once again one can define a localized and a delocalized regime, which are separated by a continuous critical curve $\lambda \mapsto \widetilde{h}_c(\lambda)$, i.e. the analog of Proposition 2.5 holds. The novelty is that from the scaling properties of Brownian motion there exists a non-negative number m_c such that $\widetilde{h}_c(\lambda) = m_c\lambda$ for every $\lambda \ge 0$. This is not all: in [5] it is shown that

$$\lim_{\lambda \searrow 0} \frac{h_c(\lambda)}{\lambda} = m_c,$$ (2.14)

a result which is more subtle than the convergence (2.12) of the free energy.

The results we have just stated, as well as their proofs, have a strong flavor of *universality*, in the sense that they are based on the idea that at small coupling (λ small) typical excursions

are very long (since the simple random walk is null recurrent), and small excursions do not contribute to the energy, so that the walk can be replaced by a Brownian motion and the sum of the charges within an excursion can be approximated by a normal variable. This does not seem to be specific of the simple random walk and binary charges and in fact the results have been shown to hold for much more general charges [17]. However, the results are (by far) not a direct consequence of standard invariance principles and the generalization to more general walks is highly non-trivial.

It would be very interesting to extend the weak-coupling results stated previously by simply assuming the validity of (2.3) with $\alpha = 1/2$. We expect in particular (2.12)–(2.14) to hold in such a generality. More generally, one would like to have the weak-coupling results for general $\alpha \in (0, 1)$. Of course the expression (2.13) has to be suitably changed, but we still expect (2.14) to hold with some m_c depending on $K(\cdot)$ only through α.

In spite of the fact that the weak-coupling results we have stated do not go yet as far as we would like, they clearly point the attention to the slope of the critical curve at the origin as a quantity of great interest. And, at least for the case $\alpha = 1/2$, this issue has been addressed in the physical literature, but without a consensus. In particular in [14] and [28] it is claimed that $m_c = 1$ (note that $m_c \leq 1$ by the upper bound in (2.11)) while in [22] and [26] it is claimed that $m_c = 2/3$ (and we know that $m_c \geq 2/3$ by the lower bound in (2.11)). For the standard copolymer model there exists numerical evidence that $2/3 < m_c < 1$, and possibly that $m_c \approx 0.83$ [8] (see also [23]), but until now there is not much clue on how to estimate this value beyond (2.11). Some of the papers we just mentioned are actually claiming that $h_c(\lambda) = h^{(m)}(\lambda)$ for every λ, with m either equal to 2/3 or 1. However only [22] deals with general disorder laws, while the others are restricted to Gaussian disorder, for which, incidentally, $h^{(m)}(\lambda) = m\lambda$.

Remark 2.8 In [27] it has been shown that the upper bound in (2.10) can be improved for all α if λ is large and $\mathbb{P}(\omega_1 < x) > 0$ for every x (which is true for instance in the case of Gaussian disorder). This is discussed briefly in Sect. 2.5. On the other hand, in Proposition 2.11 below we show also that, again for all α, we can find suitable inter-arrival distributions $K(\cdot)$ for which the lower bound in Proposition 2.6 is not optimal. These results however do not give any information on the slope of $h_c(\cdot)$ at the origin.

For the sequel of the paper it is also important to sketch the idea that leads to the lower bound in (2.10). We do this with Fig. 2 and its caption. In this strategy, the partition function is evaluated only on trajectories which are made of very long excursions with returns in *rare stretches* where the mean of the charges is atypical. This reduces the complexity of the model in two ways. On one hand, the charges act only as very rare *energetic traps* and, on the other hand, the trajectories gain only the averaged charge of these traps. A toy version of the copolymer mimicking this behavior has been introduced in [3] and it has been recently proved [4, 27] that in the strong-disorder limit the simple rare-stretch strategy of Fig. 2 identifies correctly the asymptotic behavior of its critical line. However, the numerical simulations we mentioned above show that this strategy does not fully catch the correct behavior of the copolymer model (2.2).

2.5 New results

Lower Bound: a New Rare-Stretch Strategy   A limitation in the rare-stretch strategy described in Fig. 2 is in the triviality of the behavior of the polymer in the rare advantageous stretches (the *rich blocks*). Designing a better strategy is however not so obvious. Here we
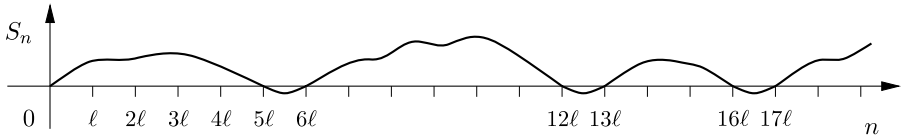
Fig. 2 The lower bound in (2.10) is proven by using a coarse-graining parameter ℓ ($1 \ll \ell \ll N$). We consider the block charges $Q_j = \sum_{n=j\ell+1}^{(j+1)\ell} \omega_n$, $j = 0, 1, 2, \ldots$, and we single out the blocks (rich blocks) for which $Q_j \leq -q\ell$ (q is a parameter that is going to be optimized). The sequence $\{Q_j\}_j$ is of course IID, so that the location of the rich blocks is simply given by a Bernoulli trial sequence with parameter $p(\ell) := \mathbb{P}(Q_1 \leq -q\ell)$. For ℓ large, $p(\ell)$ is exponentially small, so that the rich blocks are rare and typically very spaced. The lower bound in (2.10) is obtained by bounding from below the free energy. This is achieved by restricting the partition function to the trajectories S visiting the *lower half-plane* only in correspondence of the rich blocks, i.e., such that sign(S_n) < 0 if and only if n belongs to a rich block. These trajectories have very long excursions before returning to rich blocks as depicted in the drawing (the rich blocks are in this case the 5th, the 12th and the 16th). We refer to [3, 15] for a full proof of the lower bound in (2.10)

are going to present one that is ultimately going back to the original idea in [24] that a neutral environment suffices to localize the polymer (we will have to make this quantitative). The idea is therefore to look for neutral stretches, i.e. $q = 0$ (instead of negative stretches $-q < 0$), and employ a non-trivial localization strategy in these regions. The most interesting results we have been able to extract from such idea are summarized in the next statement.

Theorem 2.9 *For* $\alpha \geq 1$

$$\liminf_{\lambda \searrow 0} \frac{h_c(\lambda)}{\lambda} \geq \frac{1}{\sqrt{1+\alpha}}. \tag{2.15}$$

Moreover for $\alpha \geq 0.801$ *we have*

$$\liminf_{\lambda \searrow 0} \frac{h_c(\lambda)}{\lambda} > \frac{1}{1+\alpha}. \tag{2.16}$$

Of course these results acquire a particular interest when compared with (2.11). We want to stress that the lower bound on the slope that we are able to establish is rather explicit for all values of α (see Sect. 3, Proposition 3.2), in particular we are able to establish (2.16) for every $\alpha > \alpha_0$, with $2(1 + \alpha_0)A(\alpha_0) = 1$, where $A(\alpha)$ is the expression in the right-hand side of (3.9). The explicit bound given in the statement comes from an explicit lower bound on A, but A can be computed via numerical integration and one sees that α_0 is smaller than (but close to) 0.65. In Remark 3.3 we are going to explain that there is very little hope to make this strategy work for $\alpha = 1/2$. Nevertheless, this shows that there are better strategies than the rare stretch strategy leading to the $1/(1 + \alpha)$ bound on the slope.

Upper Bound: Fractional Moments and the Slope   The upper bound in (2.10) is just a consequence of the annealed inequality for the free energy:

$$\frac{1}{N}\mathbb{E}\log Z_{N,\omega} \leq \frac{1}{N}\log \mathbb{E}Z_{N,\omega}. \tag{2.17}$$

A natural idea to go beyond simple-minded annealing is to observe that, again thanks to Jensen's inequality, for every $\gamma > 0$

$$\frac{1}{N}\mathbb{E}\log Z_{N,\omega} \leq \frac{1}{N\gamma}\log \mathbb{E}\left[(Z_{N,\omega})^\gamma\right]. \tag{2.18}$$

Since for $\gamma = 1$ one recovers annealing (2.17) and for $\gamma \searrow 0$ (2.18) becomes an equality, it is natural to hope that non-trivial information can be obtained estimating the γ-moments of the partition function, with $0 < \gamma < 1$. This is precisely the approach which was followed in [27], and indeed it turns out that at least for λ large enough and assuming that the random variables ω_n are unbounded (more precisely, that $\mathbb{P}(\omega_1 < x) > 0$ for every x) one can prove that $h_c(\lambda) < h^{(1)}(\lambda)$ (of course, this says nothing about the critical slope). Let us also recall that in [27, Corollary 3.9] the same method allowed to prove that if $\sum_{n \geq 1} K(n) < 1$ (τ is transient, a case that we are not considering here) then $h_c(\lambda) < h^{(1)}(\lambda)$ *for every* $\lambda > 0$ and $\limsup_{\lambda \searrow 0} h_c(\lambda)/\lambda < 1$.

Two important ingredients were added in [11, 16] where, in the somewhat different context of *disordered pinning/wetting models*, it was realized first of all that it is actually sufficient to control the fractional moments of $Z_{N,\omega}$ up to N of the order of the correlation length of the annealed system, and secondly that this control can be obtained through a change-of-measure argument. Here we generalize these arguments to the case of the copolymer, and our main result is the following:

Theorem 2.10 *Choose* $K(\cdot)$ *satisfying* (2.3). *If* $\alpha > 1$

$$\limsup_{\lambda \searrow 0} \frac{h_c(\lambda)}{\lambda} < 1. \tag{2.19}$$

If $0 < \alpha \leq 1$ *there exists a positive constant* c *such that, for* $0 < \lambda \leq 1$

$$h_c(\lambda) \leq h^{(1)}\left(\lambda\left(1 - \frac{c}{|\log c\lambda^2|}\right)\right). \tag{2.20}$$

Moreover, for every $\alpha > 0$ *and* $\lambda > 0$ *one has* $h_c(\lambda) < h^{(1)}(\lambda)$.

A Further Remark on the Critical Curve   We complete the list of new results with the following one that becomes of interest in view of the various conjectures that one finds in the literature: in short, it says that in general $h_c(\cdot)$ heavily depends on the details of $K(\cdot)$ and it is certainly not simply a function of α.

Proposition 2.11 *For every* $\alpha > 0$, *every* $\lambda > 0$ *and every* $\varepsilon > 0$ *we can find* $K(\cdot)$ *that satisfies* (2.3) *such that* $h_c(\lambda) > h^{(1)}(\lambda) - \varepsilon$.

The proof is short and it is of help in understanding the result, so we give it right here.

Proof Consider the model with $\alpha = 0$ (cf. Remark 2.7, call $\widetilde{K}(\cdot)$ the particular return probability of that model). Fix $\lambda > 0$ and $\varepsilon > 0$: we have $\mathrm{F}(\lambda, h^{(1)}(\lambda) - \varepsilon) > 0$, so that, by (2.8), there exists $N \in \mathbb{N}$ such that $\mathbb{E} \log Z_{N,\omega}^{\mathrm{c}} > 0$. But such a result is unchanged if we modify the definition of $\widetilde{K}(n)$ for $n > N$, since $Z_{N,\omega}^{\mathrm{c}}$ itself is unchanged. Since (2.3) depends only on the tail of $K(\cdot)$, we are done.   □

In its simplicity, Proposition 2.11, in combination with the other results we have stated or that can be found in the literature, can be used to rule out a number of conjectures that have been made or that one may be tempted to make. For example, when the disorder is Gaussian the critical line in general is not a straight line with a slope depending only on α (recall Remark 2.8). More generally, it casts serious doubts on the fact that the critical line coincides

with $h^{(m_c)}(\lambda)$ for every λ regardless of the details of $K(\cdot)$, even if numerical evidence in [8] suggested that the critical line could coincide with $h^{(m_c)}(\cdot)$ in the particular case of the standard copolymer model.

For the remainder, we stress that we systematically develop the arguments first for $\omega_1 \sim \mathcal{N}(0, 1)$ and then give the modifications needed to deal with the general charge distributions of Definition 2.1.

3 Neutral Stretches and a Lower Bound Strategy

In this section we are going to give a proof of Theorem 2.9.

Proposition 3.1 *Let us consider the general copolymer model with $\omega_1 \sim \mathcal{N}(0, 1)$. For every $\lambda > 0$ we have*

$$h_c(\lambda) \geq \sqrt{\frac{2\mathrm{F}(\lambda, 0)}{1 + \alpha}}. \tag{3.1}$$

Proof For $\ell \in \mathbb{N}$ let us define the random variable

$$\mathrm{F}_\ell(\lambda, h; \omega) := \frac{1}{\ell} \log \mathbf{E}\left[\exp\left(-2\lambda \sum_{n=1}^{\ell} (\omega + h)\Delta_n \right); \ell \in \tau \right]. \tag{3.2}$$

We now claim that for every $\delta > 0$ we have

$$\liminf_{\ell \to \infty} \frac{1}{\ell} \log \mathbb{P}\big(\mathrm{F}_\ell(\lambda, h; \omega) \geq (1 - \delta)\mathrm{F}(\lambda, 0)\big) \geq -\frac{1}{2}h^2. \tag{3.3}$$

To see this, we first observe that if $\mathbb{P}_{\ell,h}$ is the law of $(\omega_1 - h, \omega_2 - h, \ldots, \omega_\ell - h)$, then

$$\mathcal{S}\left(\mathbb{P}_{\ell,h} | \mathbb{P}_{\ell,0}\right) := \mathbb{E}_{\ell,h}\left[\log\left(\frac{d\mathbb{P}_{\ell,h}}{d\mathbb{P}_{\ell,0}}\right)\right] = \frac{1}{2}\ell h^2. \tag{3.4}$$

We now recall the entropy inequality

$$\log\left(\frac{\mathbb{P}_{\ell,0}(E)}{\mathbb{P}_{\ell,h}(E)}\right) \geq -\frac{1}{\mathbb{P}_{\ell,h}(E)}\left(\mathcal{S}\left(\mathbb{P}_{\ell,h} | \mathbb{P}_{\ell,0}\right) + \frac{1}{e}\right), \tag{3.5}$$

which holds for arbitrary non-null events [15, Appendix A.2], and by choosing for E the event in the left-hand side of (3.3) (call it E_ℓ) we see that

$$\mathbb{P}_{\ell,h}(E_\ell) = \mathbb{P}_{\ell,0}\big(\mathrm{F}_\ell(\lambda, 0; \omega) \geq (1 - \delta)\mathrm{F}(\lambda, 0)\big), \tag{3.6}$$

where of course in the right-hand side we can write $\mathbb{P}$ instead of $\mathbb{P}_{\ell,0}$. But the existence of the infinite volume limit (2.4), together with the fact that $\mathrm{F}(\lambda, 0) > 0$ for $\lambda > 0$ (which follows from Proposition 2.6), guarantees that $\lim_{\ell \to \infty} \mathbb{P}_{\ell,h}(E_\ell) = 1$ and this, combined of course with (3.4) and (3.5), yields (3.3).

The rest of the argument follows the line of [3], alternatively see [15, Chap. 6], and it is based on chopping the sequence of charges into portions of length ℓ and checking whether the charge sub-sequence in the block is in E_ℓ (these are the *rich blocks*), namely whether $(\omega_{j\ell+1}, \omega_{j\ell+2}, \ldots) \in E_\ell$ for $j = 0, 1, \ldots$. These are of course independent events giving
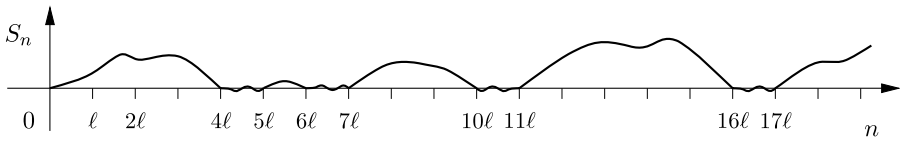
Fig. 3 The novel strategy, compare with Fig. 2, is based on targeting $q = 0$ regions, that is the (four) rich blocks are in this case the ones where $Q_\ell = o(\ell)$. This is actually implemented in the proof by a change of measure argument. More precisely, the rich blocks in the Gaussian case are the ones in which the charges look like the original charges ω shifted down of h, so that $\omega + h$ is a sequence of centered random variables. Once again the lower bound is achieved by considering trajectories that stay in the *upper half-plane* outside of the rich blocks and that touch the interface exactly at the beginning and the end of a rich block. In a rich block, however, we keep the contribution of all the trajectories, that can therefore oscillate between the two solvents in order to optimize the energetic gain

rise to a Bernoulli sequence of parameter $p(\ell) := \mathbb{P}(E_\ell)$. Once ω is chosen, the rich blocks are identified and one estimates from below $Z_{N,\omega}$ by restricting to path configurations that visit the sites $j\ell$ and $(j + 1)\ell$ for all j's for which the jth block is rich (and of course $(j + 1)\ell \leq N$) and that do not enter the lower half-plane outside of rich blocks (see Fig. 3 and its caption for more details). The free energy bound one obtains is

$$\mathrm{F}(\lambda, h) \geq p(\ell)\left[(1 - \delta)\mathrm{F}(\lambda, 0) - (1 + \alpha)\frac{h^2}{2} + o_\ell(1)\right], \tag{3.7}$$

therefore $\mathrm{F}(\lambda, h) > 0$ if the term between square brackets is positive. Since ℓ can be chosen arbitrarily large and (then) δ arbitrarily small, we obtain (3.1).   □

The result we have just stated becomes particularly effective when coupled with the next statement.

Proposition 3.2 *Choose $K(\cdot)$ that satisfies* (2.3) *and ω_1 as in Definition* 2.1.

(1) *If $\alpha \geq 1$ then*

$$\lim_{\lambda \searrow 0} \frac{1}{\lambda^2}\mathrm{F}(\lambda, 0) = \frac{1}{2}. \tag{3.8}$$

(2) *If $\alpha \in (0, 1)$ then for every $\kappa > 0$ we have*

$$\liminf_{\lambda \searrow 0} \frac{1}{\lambda^2}\mathrm{F}(\lambda, 0) \geq \frac{\kappa}{\Gamma(1 - \alpha)}\int_0^\infty \frac{\exp(-t)}{t^{1+\alpha}}\mathbb{E}\left[\log\cosh\left(z\sqrt{t/\kappa}\right)\right]\mathrm{d}t - \kappa\frac{1 - \alpha}{\alpha}, \tag{3.9}$$

where z is a standard Gaussian random variable $\mathcal{N}(0, 1)$ and $\Gamma(u) = \int_0^\infty \mathrm{d}t\, t^{u-1}\mathrm{e}^{-t}$.

Proof The case $\alpha > 1$ has been already considered in [15, Chap. 6], but we detail it here for completeness. It is slightly more intuitive in this argument to work with $\mathcal{Z}_{N,\omega}$ of (2.1) (where it is understood that $\mathrm{sign}(S_n) := 1 - 2\Delta_n$), and of course $\log(\mathcal{Z}_{N,\omega}/Z_{N,\omega}) = \exp(\lambda \sum_{n=1}^N \omega_n) = \exp(o(N))$, since in this proof $h = 0$. The fact that, for every $\alpha > 0$, $\limsup_{N\to\infty}(1/N)\mathbb{E}\log\mathcal{Z}_{N,\omega} \leq \log\mathrm{M}(\lambda)$ is an immediate consequence of Jensen's inequality and of course $\log\mathrm{M}(\lambda) \sim \lambda^2/2$ as $\lambda \searrow 0$. For what concerns the inferior limit, we still apply Jensen's inequality, but not for switching $\mathbb{E}$ and $\log$, rather $\log$ and $\mathbf{E}$. Note that, with

the notation $\omega(j, k) := \sum_{n=j+1}^{k} \omega_n$,

$$\mathcal{Z}_{N,\omega} \geq \mathbf{E}\left[\prod_{j:j\geq 1, \tau_j \leq N} \cosh\left(\lambda\omega(\tau_{j-1}, \tau_j)\right)\right], \tag{3.10}$$

which is obtained by integrating out the random signs $s = \{s_j\}_j$ (the inequality is due to neglecting the last, *incomplete*, excursion, when present). With the notation $\psi(t) := \log\cosh(t)$, we get then

$$\frac{1}{N}\mathbb{E}\log\mathcal{Z}_{N,\omega} \geq \frac{1}{N}\mathbf{E}\left[\sum_{j:j\geq 1, \tau_j \leq N}\mathbb{E}\psi\left(\lambda\omega(\tau_{j-1}, \tau_j)\right)\right] \overset{N\to\infty}{\longrightarrow} \frac{1}{\mathbf{E}[\tau_1]}\mathbf{E}\mathbb{E}\psi\left(\lambda\omega(0, \tau_1)\right), \tag{3.11}$$

where we have used also the law of large numbers. Now we observe that, by the integrability properties of ω_1, we have that for every n

$$\lim_{\lambda\searrow 0}\frac{1}{\lambda^2}\mathbb{E}\psi\left(\lambda\omega(0, n]\right) = \frac{1}{2}\mathbb{E}\left[(\omega(0, n])^2\right] = \frac{n}{2}. \tag{3.12}$$

Therefore, by applying Fatou's Lemma we infer from (3.11) that

$$\liminf_{\lambda\searrow 0}\frac{1}{\lambda^2}\mathrm{F}(\lambda, 0) \geq \frac{1}{\mathbf{E}[\tau_1]}\sum_n \frac{n}{2}K(n) = \frac{1}{2}, \tag{3.13}$$

and the proof of the case $\alpha > 1$ is complete.

The case $\alpha \leq 1$ is more delicate and one has to go beyond the direct use of Jensen's inequality. The estimates can be performed by using a change-of-measure argument via a standard entropy inequality (see e.g. [15, (A.10)]). The case $\alpha < 1$ is fully detailed in [15, (6.17)] (the proof in there follows from sharpening an argument that appears in [5]). The case $\alpha = 1$ is however not treated for the specific question we address here, so we give a proof (which is close to the proof with which one establishes (3.9)).

Set $\alpha = 1$. Given $b > 0$ define $K_b(n) = K(n)\exp(-bn)/(\sum_m K(m)\exp(-bm))$. Formula (6.13) in [15] tells us that for every $b > 0$

$$\begin{aligned}\mathrm{F}(\lambda, 0) &\geq \frac{1}{m_b}\sum_n K_b(n)\mathbb{E}\left[\psi(\lambda\omega(0, n])\right] - \frac{1}{m_b}\sum_n K_b(n)\log\left(\frac{K_b(n)}{K(n)}\right)\\ &=: e(b) - s(b),\end{aligned} \tag{3.14}$$

where $m_b = \sum_n nK_b(n)$. In [15, Proposition B.2] it is shown that $s(b) = o(b)$ (for the case we are considering here, the result is a direct consequence of $\sum_n \exp(-bn)/n \sim -\log b$, as $b \searrow 0$). Moreover by using

$$\log\cosh(x) \geq \frac{1}{2}x^2 - \frac{1}{12}x^4, \quad x \in \mathbb{R}, \tag{3.15}$$

and, by setting $b = \lambda^2$, from (3.14) we obtain

$$\frac{1}{\lambda^2}\mathrm{F}(\lambda, 0)$$

$$\geq \frac{1}{2m_{\lambda^2}}\sum_n K_{\lambda^2}(n)\mathbb{E}\left[\left(\sum_{j=1}^n \omega_j\right)^2\right] - \frac{\lambda^2}{12m_{\lambda^2}}\sum_n K_{\lambda^2}(n)\mathbb{E}\left[\left(\sum_{j=1}^n \omega_j\right)^4\right] - \frac{s(\lambda^2)}{\lambda^2}$$

$$\geq \frac{1}{2m_{\lambda^2}} \sum_n K_{\lambda^2}(n) n - \frac{c\lambda^2}{m_{\lambda^2}} \sum_n K_{\lambda^2}(n) n^2 - \frac{s(\lambda^2)}{\lambda^2}$$

$$\geq \frac{1}{2} - \frac{c'\lambda^2}{m_{\lambda^2}} \sum_n \exp(-\lambda^2 n) - \frac{s(\lambda^2)}{\lambda^2}, \tag{3.16}$$

where c is a constant that depends only on the fourth moment of ω_1 and c' comes from approximating $K(n)$ with its limit behavior and from neglecting $\sum_n K(n) \exp(-\lambda^2 n)$ in the denominator (say, for $\lambda \leq 1$). Therefore $\liminf_{\lambda \searrow 0} F(\lambda, 0)/\lambda^2 \geq 1/2$ since m_{λ^2} diverges as $\lambda \searrow 0$ and $s(\lambda^2)/\lambda^2$ tends to zero, as pointed out before. $\qquad \square$

Remark 3.3 For $\alpha = 1/2$, from a numerical estimation of the right-hand side of (3.9), we obtain that $l := \liminf_{\lambda \searrow 0} F(\lambda, 0)/\lambda^2 > 0.227$. On the other hand, one would need $l > 1/3$ for our new strategy to be better than the older one, i.e., to be able to prove that $\liminf_{\lambda \searrow 0} h_c(\lambda)/\lambda > 2/3$ with our method (just recall (3.1)). An evaluation of l by using the transfer matrix method (with the software developed in [8]) for small values of λ suggests that l is below $1/3$, even if it looks rather close to it, which in particular tells us that the neutral-stretch strategy is probably better than the old one down to α very close to $1/2$.

Proof of Theorem 2.9, Gaussian charges Formula (2.15) is an immediate consequence of Propositions 3.1 and 3.2(1). Formula (2.16), for $\alpha < 1$ but close to 1, follows from Propositions 3.1 and 3.2(2), plus the observation that the limit of the right-hand side of (3.9) as $\alpha \nearrow 1$ is equal to 1/2 (this has been already pointed out and detailed in [15, Remark 6.4]). A more quantitative bound can be obtained as follows. By using the inequality (3.15), one can bound from below the right-hand side in (3.9) by a quantity that can be explicitly computed:

$$\liminf_{\lambda \searrow 0} \frac{1}{\lambda^2} F(\lambda, 0) \geq \frac{1}{2} - \frac{1-\alpha}{4\kappa} - \kappa \left(\frac{1-\alpha}{\alpha} \right) \stackrel{\kappa = \sqrt{\alpha}/2}{=} \frac{1}{2} - \frac{1-\alpha}{\sqrt{\alpha}}, \tag{3.17}$$

where we used that $\Gamma(2 - \alpha) = (1 - \alpha)\Gamma(1 - \alpha)$. From (3.1) we see that (2.16) holds if $\liminf_{\lambda \searrow 0} F(\lambda, 0)/\lambda^2 > 1/(2(1 + \alpha))$, so that from (3.17) we have that (2.16) holds if $\alpha > 0.800981\dots$. A numerical evaluation of the full expression (3.9) shows that (2.16) holds down to $\alpha = 0.65$. $\qquad \square$

Proof of Theorem 2.9, general charges As explained in Sect. 2.4, one expects a universal behavior for $\lambda \searrow 0$ and the Gaussian bounds should remain in force. And in fact the strategy that leads to Proposition 3.1 can be generalized without much effort, but Proposition 3.1 becomes more complex to state and somewhat involved. It is therefore preferable to approach the problem under a slightly different angle, which in the end is just dealing with Propositions 3.1 and 3.2 at the same time.

The key point is to replace the change of measure used in the Gaussian case with the standard *tilting* procedure of Cramèr Large Deviation Theorem (which coincides with a shift of the mean for Gaussian variables). We therefore modify the law of $(\omega_1, \dots, \omega_\ell)$ by introducing the relative density $f_\ell(\omega) := \prod_{j=1}^\ell \exp(\mu_h \omega_j)/M(\mu_h)^\ell$, and μ_h is chosen so that $\mathbb{E}[f_\ell(\omega)\omega_1] = -h$. The target event E_ℓ in (3.3) should now be replaced by

$$E_\ell = \{\omega : F_\ell(\lambda, h; \omega) \geq (1 - \delta)\widetilde{F}_h(\lambda)\}, \tag{3.18}$$

where $\widetilde{F}_h(\lambda)$ is the free energy of the model with IID charges $\widetilde{\omega}$, with the law of $\widetilde{\omega}_1$ given by the law of ω_1 times the density $\exp(\mu_h \omega_1)/M(\mu_h)$. Note that the new charges $\widetilde{\omega}$ have

mean $-h$ and, in general, their variance is not equal to one, but it converges to 1 as h vanishes. This induces a number of changes that lead to replacing (3.7) by

$$\mathrm{F}(\lambda, h) \geq p(\ell) \left[(1 - \delta) \widetilde{\mathrm{F}}_h(\lambda) - (1 + \alpha) \Sigma(-h) + o_\ell(1) \right], \tag{3.19}$$

where $\Sigma(\cdot)$ is the Cramèr functional of the law of ω_1, i.e. the Legendre transform of $\mathrm{M}(\cdot)$. It is well known that $\Sigma(-h) \sim h^2/2$ as $h \searrow 0$, thus for every $\epsilon > 0$ there exists $h_\epsilon > 0$ such that $(\lambda, h) \in \mathcal{L}$ if $h < h_\epsilon$ and

$$(1 - \delta) \widetilde{\mathrm{F}}_h(\lambda) \geq (1 + \alpha)(1 + \epsilon) \frac{h^2}{2}. \tag{3.20}$$

At this point, $\widetilde{\mathrm{F}}_h(\lambda)$ can be bounded from below precisely as it is done in Proposition 3.2. Choosing $h = m\lambda$, the argument in that case always leads to estimating the moments (in fact, the second and the fourth moments suffice) of ω_1, that has to be replaced with the centered variable $\widetilde{\omega}_1 - h$ of variance $1 + O(\lambda)$ (and $\mathbb{E}[(\widetilde{\omega}_1 + h)^4] = \mathbb{E}[\omega_1^4](1 + O(\lambda))$).

This completes the proof of Theorem 2.9. $\qquad\square$

4 Fractional Moments and Upper Bounds

In this section we prove Theorem 2.10.

We use the short-cut notation $Z_N := Z_{N,\omega}^c$ and $Z_{a,b} := Z_{(b-a),\theta^a\omega}^c$ for $a < b$, where θ is the shift operator such that $(\theta\omega)_n = \omega_{n+1}$. We start by pointing out that, by integrating out the $\{s_j\}_j$ variables, we can write

$$Z_N = \mathbf{E}\left[\prod_{j:\tau_j \leq N} \varphi\big(\lambda\omega(\tau_{j-1}, \tau_j] + \lambda h(\tau_j - \tau_{j-1})\big); N \in \tau \right], \tag{4.1}$$

with $\varphi(t) = (1 + \exp(-2t))/2$ and $\omega(j, k] := \sum_{n=j+1}^{k} \omega_n$.

One of the two key ingredients for the proof is the following decomposition of the partition function, which is just based on partitioning the space of trajectories according to the location of the first point of τ after k (call it j) and the last point before k (call it i):

$$Z_N = \sum_{j=k}^{N} Z_{j,N} \sum_{i=0}^{k-1} K(j-i)\varphi\big(\lambda\omega(i, j] + \lambda h(j-i)\big) Z_i. \tag{4.2}$$

The second key ingredient is the use of fractional moments, so we set for $0 < \gamma < 1$

$$A_N := \mathbb{E}\left[(Z_N)^\gamma \right], \tag{4.3}$$

(we do not make the γ-dependence explicit in the notation). From (4.2) and the basic inequality

$$\left(\sum a_i \right)^\gamma \leq \sum a_i^\gamma, \tag{4.4}$$

that holds whenever $a_i \geq 0$ for every i if $0 < \gamma < 1$, we obtain

$$A_N \leq \sum_{j=k}^{N} A_{N-j} \sum_{i=0}^{k-1} B(j-i) A_i, \tag{4.5}$$

where

$$B(j) := K(j)^\gamma \mathbb{E}\big[\big(\varphi\big(\lambda\omega(0,j] + \lambda hj\big)\big)^\gamma\big]. \tag{4.6}$$

For later use we point out that, again thanks to (4.4),

$$B(j) \leq K(j)^\gamma 2^{-\gamma} \big[\exp\big(j\big(\log M(-2\gamma\lambda) - 2\gamma\lambda h\big)\big) + 1\big]. \tag{4.7}$$

Lemma 4.1 *If there exist* $\gamma \in (0,1)$ *and* $k \in \mathbb{N}$ *such that*

$$\sum_{j=k}^\infty \sum_{i=0}^{k-1} B(j-i)A_i \leq 1, \tag{4.8}$$

then $\sup_N A_N < \infty$.

Proof Combining (4.5) and the hypothesis (4.8) one readily obtains for every $N \geq k$

$$A_N \leq \max_{j=0,\dots,N-k} A_j. \tag{4.9}$$

If one sets $A_k^\star := \max_{j=0,\dots,k-1} A_j$, then from (4.9) one has $A_N \leq A_k^\star$ for every N. $\qquad\square$

Remark 4.2 Note that if $\sup_N A_N < \infty$ one has

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}\big[\log Z_N\big] \leq \lim_{N\to\infty} \frac{1}{\gamma N}\log A_N = 0 \tag{4.10}$$

and therefore $\mathrm{F}(\lambda, h) = 0$.

4.1 Gaussian Disorder and $\alpha > 1$

Let us set $\alpha > 1$ and $\omega_1 \sim \mathcal{N}(0,1)$, and fix $\lambda_0 < \infty$. We are going to show that if $\rho < 1$ is sufficiently close to one, $\mathrm{F}(\lambda, \rho\lambda) = 0$ for $\lambda \leq \lambda_0$. To this purpose we fix $\gamma < 1$ such that $\gamma(1+\alpha) > 2$ and notice that

$$\log M(-2\gamma\lambda) - 2\gamma\rho\lambda^2 = 2\gamma\lambda^2(\gamma - \rho) < 0, \tag{4.11}$$

provided that $\rho > \gamma$. Therefore, thanks also to (4.8) and (4.7), it is sufficient to show that

$$\sum_{j=k}^\infty \sum_{i=0}^{k-1} K(j-i)^\gamma A_i \leq 2^{\gamma-1}. \tag{4.12}$$

Computing the sum over j, we see that there exists $C > 0$ depending on $K(\cdot)$ such that the left-hand side in (4.12) is smaller than

$$C\sum_{i=0}^{k-1} \frac{A_i}{(k-i)^{\gamma(\alpha+1)-1}}. \tag{4.13}$$

In view of the last two formulas we see that it is sufficient to show that there exist $\rho \in (\gamma, 1)$ such that for $\lambda \leq \lambda_0$ and $h = \rho\lambda$ it is possible to find $k \in \mathbb{N}$ such that

$$\sum_{i=0}^{k-1} \frac{A_i}{(k-i)^{\gamma(\alpha+1)-1}} \leq c_1 := \frac{2^{\gamma-1}}{C}. \tag{4.14}$$

We choose k to be equal to the integer part of $1/(\lambda^2(1-\rho))$ (but there is no loss of generality in choosing $1/(\lambda^2(1-\rho)) \in \mathbb{N}$, so we do that) and we note that k is large (uniformly in $\lambda \leq \lambda_0$) if ρ is close to 1. Hence, by Jensen's inequality we have

$$A_i \leq [\mathbb{E}Z_i]^\gamma \leq \exp\left(2i(1-\rho)\lambda^2\gamma\right) \leq e^{2\gamma}, \tag{4.15}$$

for $i \leq k$. Therefore, if R is an integer number smaller than k

$$\sum_{i=0}^{k-R} \frac{A_i}{(k-i)^{\gamma(\alpha+1)-1}} \leq e^{2\gamma} \sum_{l=R}^{\infty} \frac{1}{l^{\gamma(\alpha+1)-1}}. \tag{4.16}$$

The right-hand side can be made arbitrarily small by choosing R large (of course this requires k to be large, but this is again the requirement of choosing ρ close to 1), because γ is such that $\gamma(\alpha+1) > 2$. We choose R large enough so that such an expression is smaller that $c_1/2$. It suffices now to prove that

$$\sum_{i=k-R+1}^{k-1} \frac{A_i}{(k-i)^{\gamma(\alpha+1)-1}} \leq c_1/2, \tag{4.17}$$

for ρ sufficiently close to 1. In analogy with [11], we are going to shift the random variables ω_i in order to bound A_i. We define the shifted charges by introducing the new law $\mathbb{P}_{N,y}$

$$\frac{d\mathbb{P}_{N,y}}{d\mathbb{P}}(\omega) = \exp\left(y\sum_{i=1}^{N}\omega_i - Ny^2/2\right), \tag{4.18}$$

and control A_i by applying the Hölder inequality with $p = 1/\gamma$ and $q = 1/(1-\gamma)$:

$$A_N = \mathbb{E}_{N,y}\left[(Z_N)^\gamma \frac{d\mathbb{P}}{d\mathbb{P}_{N,y}}(\omega)\right] \leq \left(\mathbb{E}_{N,y}[Z_N]\right)^\gamma \exp\left(\frac{\gamma}{2(1-\gamma)}y^2N\right). \tag{4.19}$$

We apply (4.19) with $y = \lambda(1-\rho)^{1/2} = 1/\sqrt{k}$, $N = i$ and $k - R < i < k$. Note that with this choice the exponential term in the right-most side of (4.19) is bounded by the constant $\exp(\gamma/(2(1-\gamma)))$, that $\mathbb{E}_{i,y}[Z_i]$ coincides with

$$\mathbf{E}\left[\exp\left(2\lambda^2\left((1-\rho) - (1-\rho)^{1/2}\right)\sum_{n=1}^{i}\Delta_n\right); i \in \tau\right] \tag{4.20}$$

and that for ρ close to 1

$$2\lambda^2\left((1-\rho) - (1-\rho)^{1/2}\right) \leq -\lambda^2(1-\rho)^{1/2}. \tag{4.21}$$

Remarking that

$$\min_{k-R<i<k} i\lambda^2(1-\rho) \geq 1/2, \tag{4.22}$$

for k large, we see that the quantity in (4.20) can be made arbitrarily small, uniformly in $\lambda \leq \lambda_0$ and $k - R < i < k$ by choosing ρ suitably close to 1, because

$$\lim_{N\to\infty} \mathbf{E}\left[\exp\left(-\frac{q}{N}\sum_{n=1}^{N}\Delta_n\right)\right] = \exp(-q/2), \tag{4.23}$$

which follows from the Dominated Convergence Theorem since $(1/N)\sum_{n=1}^{N}\Delta_n$ tends almost surely to $1/2$. The latter statement follows by observing that if we set $Y_N := \max\{n : \tau_n \leq N\}$ (number of renewals up to N), by the law of large numbers we have that Y_N/N tends a.s. to $1/\mathbf{E}[\tau_1]$ as N tends to infinity and

$$\frac{1}{N}\sum_{n=1}^{N}\Delta_n \geq \frac{Y_N}{N}\cdot\frac{1}{Y_N}\sum_{j=1}^{Y_N}(\tau_j - \tau_{j-1})\mathbf{1}_{s_j=-1}. \tag{4.24}$$

Since $\{(\tau_j - \tau_{j-1})\mathbf{1}_{s_j=-1}\}_{j=1,2,...}$ is an IID sequence and $\mathbf{E}[\tau_1\mathbf{1}_{s_1=-1}] = \mathbf{E}[\tau_1]/2$, again by the law of large numbers the right-hand side in (4.24) converges almost surely to $1/2$. We can reverse the inequality in (4.24) by summing over j up to $Y_N + 1$, so that (4.23) is proven.

On the other hand

$$\sum_{i=k-R+1}^{k-1}\frac{A_i}{(k-i)^{\gamma(\alpha+1)-1}} \leq \max_{k-R<j<k}A_j\sum_{i=1}^{\infty}\frac{1}{i^{\gamma(\alpha+1)-1}}, \tag{4.25}$$

so that, since the sum converges, the right-hand side can be made arbitrarily small, hence smaller than $c_1/2$, and we are done.

The fact that $h_c(\lambda) < h^{(1)}(\lambda)$ for every $\lambda > 0$ is a direct consequence of the fact that λ_0 is arbitrary.

4.2 Gaussian disorder and $\alpha < 1$

Again, fix $\lambda_0 < \infty$. We let $h = h(\lambda) = \lambda(1 - \frac{c}{|\log c\lambda^2|})$ and we aim at proving (4.8) for some $c > 0$, uniformly in $0 < \lambda \leq \lambda_0$. We choose k to be equal to (the integer part of) $\frac{|\log c\lambda^2|}{c\lambda^2}$ and $\gamma = 1 - (\log k)^{-1}$. Notice that k can be made arbitrarily large (and therefore γ close to 1) uniformly for all $\lambda \leq \lambda_0$ choosing c small enough. Therefore, we will consider that k is large when we need to.

We use (4.7) to find a simple bound on $B(\cdot)$. Since

$$\log(M(-2\gamma\lambda)) - 2\gamma\lambda h(\lambda) = 2\gamma\lambda^2\left(\gamma - \frac{h(\lambda)}{\lambda}\right) \leq 0, \tag{4.26}$$

when c is well chosen, we have

$$B(j) \leq K(j)^{\gamma}2^{1-\gamma}. \tag{4.27}$$

Therefore the condition (4.8) will be fulfilled if we can show that

$$\sum_{j=k}^{\infty}\sum_{i=0}^{k-1}\frac{A_i}{(j-i)^{(\alpha+1)\gamma}} \leq c_2 \tag{4.28}$$

for a suitable constant c_2 which is independent of c or λ. First of all we get rid of γ in the denominator as follows:

$$\sum_{j=k}^{\infty}\frac{1}{(j-i)^{(\alpha+1)\gamma}} = \sum_{j=(k-i)}^{\infty}j^{-(\alpha+1)\gamma} \tag{4.29}$$

$$\leq \sum_{j=(k-i)}^{k^6} j^{-(\alpha+1)} \exp\left(\frac{(\alpha+1)\log j}{\log k}\right) + \sum_{j=k^6+1}^{\infty} j^{-(1+\frac{\alpha}{2})} \leq c_3(k-i)^{-\alpha},$$

(4.30)

for some constant $c_3 < \infty$, provided c is such that $(\alpha+1)\gamma \geq 1 + (\alpha/2)$. Hence (4.28) will be satisfied if

$$\sum_{i=0}^{k-1} \frac{A_i}{(k-i)^\alpha} \leq \frac{c_2}{c_3}.$$

(4.31)

To estimate A_i we use (4.19) with $y = \sqrt{c}\lambda/|\log c\lambda^2|$, $N = i$ and $i < k$. With these settings, one can check that the exponential term in the right-hand side of (4.19) is bounded by a constant c_4 that does not depend on λ or c (and therefore will be harmless), and that $\mathbb{E}_{i,y}[Z_i]$ coincides with

$$\mathbf{E}\left[\exp\left(2\frac{\lambda^2}{|\log c\lambda^2|}(c-\sqrt{c})\sum_{n=1}^{i}\Delta_n\right); i \in \tau\right] \leq \mathbf{E}\left[\exp\left(-\frac{i}{\sqrt{c}k}\frac{\sum_{n=1}^{i}\Delta_n}{i}\right); i \in \tau\right]$$

(4.32)

for c small enough. The last expression is in any case smaller than $\mathbf{P}[i \in \tau]$, which is itself bounded above by $c_5 i^{\alpha-1}$ for some constant c_5 depending on $K(\cdot)$ (see [12, Theorem B]).

We need a better upper bound for large i. This is provided by:

Lemma 4.3 *Assume that $0 < \alpha \leq 1$. Then,*

$$\lim_{q\to\infty} \limsup_{N\to\infty} \mathbf{E}\left[\exp\left(-\frac{q}{N}\sum_{n=1}^{N}\Delta_n\right)\middle| N \in \tau\right] = 0.$$

(4.33)

We fix a small $a > 0$, and consider $ak + 1 \leq i < k$. Since therefore $i/k > a$, thanks to Lemma 4.3 one deduces that, if one chooses c sufficiently small (how small, depending on a), the quantity in the right-hand side of (4.32) is bounded above by $a\mathbf{P}[i \in \tau]$.

We can summarize our result concerning A_i as follows:

$$A_i \leq \begin{cases} (\mathbf{P}(i \in \tau))^\gamma \leq c_6 i^{\alpha-1} & \text{for } i \leq ak, \\ (a\mathbf{P}(i \in \tau))^\gamma \leq c_6 a^\gamma i^{\alpha-1} \leq c_6 a^{\gamma+\alpha-1} k^{\alpha-1} & \text{for } ak+1 \leq i \leq k-1, \end{cases}$$

(4.34)

where in both cases we used that $i^{1-\gamma} \leq c_7$ for $i \leq k$. Hence

$$\sum_{i=0}^{ak} \frac{A_i}{(k-i)^\alpha} \leq c_8 a^\alpha \quad \text{and} \quad \sum_{i=ak+1}^{k-1} \frac{A_i}{(k-i)^\alpha} \leq c_9 a^{\gamma+\alpha-1},$$

(4.35)

where we just used the previous inequalities to estimate A_i. Since a can be chosen arbitrarily small, (4.31) is satisfied.

Proof of Lemma 4.3 for $0 < \alpha < 1$ First of all, we claim that it is sufficient to prove (4.33) for the unconditioned measure, i.e., that

$$\lim_{q\to+\infty} \limsup_{N\to\infty} \mathbf{E}\left[\exp\left(-q\frac{\sum_{n=1}^{N}\Delta_n}{N}\right)\right] = 0.$$

(4.36)

Indeed, with $X_N := \max\{n = 0, 1, \ldots, N/2 : n \in \tau\}$ (last renewal epoch up to $N/2$), one has

$$\mathbf{E}\left[\exp\left(-q\frac{\sum_{n=1}^{N}\Delta_n}{N}\right)\middle|\, N \in \tau\right] \le \mathbf{E}\left[\exp\left(-q\frac{\sum_{n=1}^{N/2}\Delta_n}{N}\right)\middle|\, N \in \tau\right]$$

$$= \sum_{k=0}^{N/2}\mathbf{E}\left[\exp\left(-q\frac{\sum_{n=1}^{N/2}\Delta_n}{N}\right)\middle|\, X_N = k\right]\mathbf{P}\left(X_N = k\middle|\, N \in \tau\right) \qquad (4.37)$$

where we used the renewal property in the second step. Next, it is not difficult to see that for every $k = 0, 1, \ldots, N/2$

$$\mathbf{P}\left(X_N = k\middle|\, N \in \tau\right) \le c_{10}\mathbf{P}\left(X_N = k\right) \qquad (4.38)$$

(this is detailed for instance in the proof of [11, Lemma 4.1]) and the claim follows.

To show (4.36), note that, again with the notation $Y_N := \max\{n : \tau_n \le N\}$, for every $\varepsilon \in (0, 1)$ we have

$$\mathbf{P}\left(\frac{Y_N}{N^\alpha} \le \varepsilon\right) = \mathbf{P}\left(\tau_{\varepsilon N^\alpha} \ge N\right) = \mathbf{P}\left(\frac{\tau_{\varepsilon N^\alpha}}{(\varepsilon N^\alpha)^{1/\alpha}} \ge \varepsilon^{-1/\alpha}\right) \overset{N\to\infty}{\longrightarrow} G_\alpha(\varepsilon^{-1/\alpha}), \qquad (4.39)$$

where we have assumed $\varepsilon N^\alpha \in \mathbb{N}$ and $G_\alpha(\cdot)$ is the integrated tail probability function of an α-stable variable, see [13, Theorems 1 and 2, pp. 448–449] from which we extract also that $G_\alpha(\varepsilon^{-1/\alpha}) \overset{\varepsilon\searrow 0}{\sim} \varepsilon C_K / \Gamma(1 - \alpha)$ (recall (2.3)). Therefore

$$\mathbf{P}\left(\frac{Y_N}{N^\alpha} \le \varepsilon\right) \le \frac{2C_K\varepsilon}{\Gamma(1 - \alpha)}, \qquad (4.40)$$

for N sufficiently large and $\varepsilon \in (0, 1)$. On the other hand

$$\mathbf{E}\left[\exp\left(-\frac{q}{N}\sum_{n=1}^{N}\Delta_n\right)\right] \le \mathbf{E}\left[\prod_{i=1}^{Y_N}\left(\frac{\exp(-q(\tau_i - \tau_{i-1})/N) + 1}{2}\right)\right]$$

$$\le \mathbf{E}\left[\prod_{i=1}^{\varepsilon N^\alpha}\left(\frac{\exp(-q(\tau_i - \tau_{i-1})/N) + 1}{2}\right)\right] + \mathbf{P}\left(\frac{Y_N}{N^\alpha} \le \varepsilon\right)$$

$$\le \mathbf{E}\left[\frac{\exp(-q\tau_1/N) + 1}{2}\right]^{\varepsilon N^\alpha} + 2C_K\varepsilon/\Gamma(1 - \alpha). \qquad (4.41)$$

Observe that

$$1 - \mathbf{E}\left[\frac{\exp(-q\tau_1/N) + 1}{2}\right] \overset{N\to\infty}{\sim} \frac{C_K q^\alpha}{2\alpha N^\alpha}\Gamma(1 - \alpha), \qquad (4.42)$$

so we have that for some constant $c_{11}(\alpha)$ (which depends only on α)

$$\limsup_{N\to\infty}\mathbf{E}\left[\exp\left(-\frac{q}{N}\sum_{n=1}^{N}\Delta_n\right)\right] \le \exp\left(-c_{11}(\alpha)(C_K\varepsilon)q^\alpha\right) + \frac{2C_K\varepsilon}{\Gamma(1 - \alpha)}$$

$$\le 4\frac{(\log q)^2}{q^\alpha\Gamma(1 - \alpha)}, \qquad (4.43)$$

where in the last step we have chosen $C_K \varepsilon = (\log q)^2/q^\alpha$ and we assumed that $q \geq q_0(\alpha)$ with $q_0(\alpha)$ is sufficiently large. The proof of Lemma 4.3 for $0 < \alpha < 1$ is therefore complete. □

4.3 Gaussian Disorder and $\alpha = 1$

The proof is very similar to that of the case $\alpha < 1$, and therefore we point out only the necessary modifications. No changes are needed up to formula (4.32) and, again, we let a be a small positive number. To avoid repetitions, it will be understood that c is chosen sufficiently small (how small depends on a), so that $1/k$ and $1 - \gamma = 1/\log k$ can be made arbitrarily small with a fixed. Using simply the fact that $A_i \leq c_{12}$, we obtain in analogy with the first bound of (4.35)

$$\sum_{i=0}^{ak} \frac{A_i}{(k-i)} \leq 2c_{12}a. \tag{4.44}$$

As for the values $i > ak$, we use the fact that (see [15, Theorem A.6])

$$\mathbf{P}(N \in \tau) \overset{N \to \infty}{\sim} \frac{c_{13}}{\log N} \tag{4.45}$$

and we claim that Lemma 4.3 still holds for $\alpha = 1$ (this will be proven in a while) so that, in analogy with (4.34), $A_i \leq a^\gamma \mathbf{P}(i \in \tau)^\gamma$ for all $ak < i < k$. Then, via (4.45) and recalling that $\gamma = 1 - 1/\log k$,

$$\sum_{i=ak+1}^{k-1} \frac{A_i}{(k-i)} \leq c_{14} \log k \max_{ak < i < k} A_i \leq c_{15}a^\gamma \tag{4.46}$$

where we used also the fact that, choosing c small enough, we can assume $\log k/(\log ak) < 2$. This concludes the proof since (4.31) is satisfied if a is small.

Proof of Lemma 4.3 for $\alpha = 1$ Again, it is sufficient to prove the claim for the unconditioned measure, i.e., to show (4.36). If we define the event

$$E_N := \left\{ (\tau_i - \tau_{i-1}) \leq \frac{N}{\sqrt{\log N}} \text{ for every } i \leq Y_N + 1 \right\}, \tag{4.47}$$

we have

$$\mathbf{E}\left[\exp\left(-q \frac{\sum_{n=1}^{N} \Delta_n}{N} \right) \right] \leq \mathbf{E}\left[\prod_{i \leq Y_N} \left(\frac{1 + e^{-q(\tau_i - \tau_{i-1})/N}}{2} \right) \right]$$

$$\leq \mathbf{E}\left[e^{-\frac{q}{4N} \tau_{Y_N}} \mathbf{1}_{\{E_N\}} \right] + \mathbf{P}\left(E_N^c \right), \tag{4.48}$$

for N sufficiently large (we simply used that $\max_{i \leq Y_N}(\tau_i - \tau_{i-1})/N$ tends to 0 for $N \to \infty$ if E_N is realized). Note also that, by the definition of E_N, $\tau_{Y_N}/N \geq 1/2$ for N large if E_N is realized. Therefore,

$$\limsup_{N \to \infty} \mathbf{E}\left[\exp\left(-q \frac{\sum_{n=1}^{N} \Delta_n}{N} \right) \right] \leq e^{-\frac{q}{8}} + \limsup_{N \to \infty} \mathbf{P}\left(E_N^c \right). \tag{4.49}$$

To show that the probability of E_N^c tends to zero with N, we start by observing that from the tail behavior of $K(\cdot)$ it follows that

$$\mathbf{P}\left(\text{there exists } i \leq \frac{N}{(\log N)^{3/4}} \text{ such that}(\tau_i - \tau_{i-1}) \geq \frac{N}{\sqrt{\log N}}\right) \leq \frac{c_{16}}{(\log N)^{1/4}}. \quad (4.50)$$

On the other hand,

$$\lim_{N\to\infty} \mathbf{P}\left(Y_N \geq \frac{N}{(\log N)^{3/4}}\right) \leq \frac{(\log N)^{3/4}}{N} \mathbf{E}[Y_N] = \frac{(\log N)^{3/4}}{N} \sum_{i=1}^{N} \mathbf{P}(i \in \tau)$$

$$\leq \frac{c_{17}}{(\log N)^{1/4}}, \quad (4.51)$$

where we used (4.45). From (4.50)–(4.51) we directly see that $\lim_{N\to\infty} \mathbf{P}(E_N^c) = 0$ and the proof is complete. $\qquad\square$

4.4 Proof of Theorem 2.10: the General Case

Once again, going beyond the case of Gaussian disorder requires only a bit of care in some steps. In both case ($\alpha > 1$ and $\alpha \leq 1$), we have to prove

$$h_c(\lambda) \leq h^{(1)}(\rho\lambda) \quad \text{for every } \lambda \leq \lambda_0, \quad (4.52)$$

where ρ is a fixed constant, chosen close to 1, in the case $\alpha > 1$ (we can get the slope result from it then), and $\rho = \rho(\lambda) = 1 - \frac{c}{|\log c\lambda^2|}$ for a small c for the case $\alpha \leq 1$. We take here a quick look at what needs to be changed from the proof of the Gaussian case.

Precisely (4.11), (4.26) have to be replaced by the observation that

$$\log M(-2\gamma\lambda) - 2\gamma\lambda h^{(1)}(\rho\lambda) = 2\gamma\lambda\left[h^{(1)}(\gamma\lambda) - h^{(1)}(\rho\lambda)\right], \quad (4.53)$$

and that the right-hand side is negative if $\rho > \gamma$ because $h^{(1)}(\cdot)$ is increasing. For what concerns instead (4.15) (and the same bound has to be used for (4.20) and (4.32)) the bound one has to use are

$$\log M(-2\lambda) - 2\lambda h^{(1)}(\rho\lambda) = \log M(-2\lambda) - \frac{1}{\rho} \log M(-2\rho\lambda)$$

$$\leq (1 - \rho) \max_{\varrho \in [\rho, 1]} \left|\frac{d}{d\varrho} g_\lambda(\varrho)\right|, \quad (4.54)$$

where $g_\lambda(\varrho) := \varrho^{-1} \log M(-2\lambda\varrho)$. Since $\lambda \in (0, \lambda_0]$, one directly verifies that the rightmost term in (4.54) is bounded by $C(\lambda_0)\lambda^2$, where $C(\lambda_0)$ is a positive constant.

Another point in which the Gaussian character of the disorder enters is in the *shifting procedure* of (4.18) and (4.19) that gives (4.20) and (4.32). The shift has to be replaced by a *tilt* and one reduces to estimates that, except for constants that depend on the law of the disorder, are the same as in the Gaussian case (these steps are fully detailed for another model in [11, Appendix A.1 and Sect. 3] and it is not very difficult to adapt them to this proof). $\qquad\square$

Note Added in Proof Theorem 2.9 can be easily improved for $\alpha > 3$. In fact $\liminf_{\lambda \searrow 0} h_c(\lambda)/\lambda \geq 1/2$ if $\alpha > 1$ simply because

$$\frac{1}{N}\mathbb{E}\log Z_{N,\omega} \quad \geq \quad -h\lambda + \frac{1}{N}\mathbf{E}\sum_{\substack{j:j\geq 1 \\ \tau_j \leq N}}\mathbb{E}[\psi(\lambda h(\tau_j - \tau_{j-1}) + \lambda\omega(\tau_{j-1}, \tau_j])]$$

$$\overset{N\to\infty}{\to} -h\lambda + \frac{1}{\mathbf{E}[\tau_1]}\mathbf{E}\mathbb{E}\psi(h\lambda\tau_1 + \lambda\omega(0, \tau_1]),$$

(which is analogous to (3.11)) and because (3.12) still holds with ($\psi(\lambda\omega(0, n])$ replaced by $\psi(\lambda hn + \lambda\omega(0, n])$, if $h = ((\lambda)$.

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